Skein relations for the link invariants coming from exceptional Lie algebras

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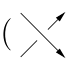
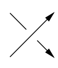
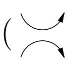
Abstract

Pulling back the weight systems associated with the exceptional Lie algebras and their standard representations by a modification of the universal Vassiliev-Kontsevich invariant yields link invariants; extending them to coloured 3-nets, we derive for each of them a skein relation.

0 Introduction

There is a well-known technique for the construction of Vassiliev link invariants: define a weight system (i.e. a linear form on the space of chord diagrams respecting certain relations) on the basis of some Lie algebraic data and pull it back by the universal Vassiliev-Kontsevich invariant. But unfortunately, the latter is not known explicitly enough to allow direct evaluation of these link invariants.

Efforts have been made to handle the universal Vassiliev-Kontsevich invariant by considering only "elementary" parts of links into which any link may be cut. This approach has been successful in so far as one may hope to find skein relations for the link invariants coming from Lie algebras—a skein relation is an equation implying a recursive algorithm for the computation of a link invariant, for example the following equation, which determines the famous Jones polynomial up to normalization:

$t^2P($  $) - t^{-2}P($  $) = (t^{-1} - t)P($  $).$

It has been shown that the link invariants obtained from the classical simple Lie algebras $\mathfrak{sl}_n$, $\mathfrak{so}_n$, and $\mathfrak{sp}_n$ satisfy certain versions of the skein relation of the HOMFLY polynomial ($\mathfrak{sl}_n$; see [LM 1]) resp. the Kauffman polynomial ($\mathfrak{so}_n$, $\mathfrak{sp}_n$; see [LM 2]). But what about the exceptional simple Lie algebras?

In [BS], we have outlined a strategy for establishing skein relations and dealt with the case of the exceptional Lie algebra $\mathfrak{g}_2$. At the price of further generalizations of the notion of links—besides branchings, we have to introduce a colouring—we can now present skein relations for all the invariants coming from exceptional simple Lie algebras

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and their standard representations. Unfortunately, we have not been able to determine all the initial values of the recursive algorithms implied by the skein relations.

To the reader who is not familiar with Lie theory, we recommend [H] and [FH]. For an introduction to Vassiliev invariants and weight systems, see [BN 1]; a more general definition of weight systems is given in [V], section 6.

Overview over the categories and functors appearing in this paper:

$$\begin{tikzcd} \mathcal{T}_{topfc} \arrow[r, "\hat{Z}_f"] \arrow[rr, bend left, "I_{(\mathfrak{g},V)}"] & \hat{\mathcal{D}}_{3c} \arrow[r, "\Psi_{\mathfrak{g}}"] & \mathcal{C}(\mathfrak{g}) \end{tikzcd}$$

the $(\mathfrak{g}, V)$-invariant

the category of **t**rivalent, **o**riented, **p**aranthesized, **f**ramed, **c**oloured tangles	the universal Vassiliev-Kontsevich invariant	the category of coloured 3-diagrams (a generalization of chord diagrams)	the $(\mathfrak{g}, V)$-weight system	a subcategory of the category of representations of $\mathfrak{g}$

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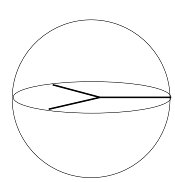
1 Invariants for coloured oriented 3-tangles

In this section, we generalize the terms 3-nets, 3-tangles, and 3-diagrams introduced in [BS] by adding orientations and colourings, and we adapt the universal Vassiliev-Kontsevich and the $(\mathfrak{g}, V)$-weight system to these generalizations. This will allow us to obtain for each exceptional Lie algebra a skein relation for a corresponding link invariant.

A 3-net is something like a "link with branchings". To describe the situation near a trivalent vertex (i.e. near a branching point), we will need the following notion:
Let B be the open unit ball in $\mathbf{R}^3$, i.e. $\{x \in \mathbf{R}^3; |x| < 1\}$, together with the distinguished subset $T := \{(t,0,0)|0 \leq t \leq 1\} \cup \{(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)t|0 \leq t \leq 1\} \cup \{(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)t|0 \leq t \leq 1\}$.

B =



Definition 1.1 *An* oriented framed 3-net *is a subset N of $\mathbf{R}^3$ with a finite subset $\{t_1, \ldots, t_n\} \subset N$ such that:*

(i) there exist disjoint open subsets $U_1, \ldots, U_n$ and diffeomorphisms $f_i : U_i \to B$ ($i = 1, \ldots, n$) such that U_i is a neighbourhood of t_i, $f_i(t_i) = (0,0,0)$, and $f_i(N \cap U_i) = T$,

(ii) $\tilde{N} := N \backslash \left(\bigcup_{i=1}^{n} f_i^{-1}(\{x \in B; |x| < \frac{1}{2}\}) \right)$ *is an embedded smooth closed compact 1-dimensional manifold,*

together with:

(iii) an orientation on $N \backslash \{t_1, \ldots, t_n\}$,

(iv) a smooth vector field on N *that is nowhere tangent to* N *(and in particular nowhere zero) .*

The points $t_1, \ldots, t_n$ *are called* trivalent vertices *of* N*; boundary points* x *of* $\tilde{N}$ *with* $x \notin U_i$ $(\forall i)$ *are called* univalent vertices *of* N*. An* edge *of* N *is a connected component of* $N \backslash \{t_1, \ldots, t_n\}$.

Definition 1.2 *Let* $\mathcal{C}ol := \{so(lid), da(shed), do(tted)\}$ *be the set of colours.*

Definition 1.3 *An* oriented framed coloured 3-net *is an oriented framed 3-net together with a colouring of the edges, i.e. a map* $col : \{edges\ of\ N\} \to \mathcal{C}ol$.

The definitions for *equivalent, closed*, and *planar* oriented framed coloured 3-nets and 3-tangles are the adapted versions of the corresponding definitions for framed 3-nets and 3-tangles in [BS].

The category $\mathcal{T}_{topfc}$ of trivalent oriented parenthesized framed coloured tangles is the coloured and oriented analogon of the category $\mathcal{T}_{tpf}$ in [BS]. Its objects are non-associative coloured words:

Definition 1.4 *A* non-associative coloured word *is a word* w *in the alphabet* $\{), (\} \cup \{c^{\to}, c^{\leftarrow} | c \in \mathcal{C}ol\}$*, such that* w *is equal to the empty word or to* $(c^{\to}), (c^{\leftarrow})$*, or* $(w_1 w_2)$ *where* c *is any colour and* w_1, w_2 *are non-associative coloured words. For every non-associative coloured word* w*, we identify* (w) *with* w*.*
For $x \in \{\to, \leftarrow\}$ *let* $\overline{x}$ *be* $\leftarrow$ *if* $x = \to$ *and* $\to$ *if* $x = \leftarrow$.
The underlying string $u(w)$ of a non-associative coloured word w *is the sequence in* $c^{\to}, c^{\leftarrow}$ *with* $c \in \mathcal{C}ol$ *one obtains from* w *by omitting all parentheses.*
The length $l(w)$ *of a non-associative coloured word is the number of symbols in* $u(w)$.

Example 1.5

$$u\left((((c_1^{\to})(c_2^{\leftarrow}))(((c_3^{\leftarrow})(c_2^{\to}))(c_1^{\leftarrow})))\right) = c_1^{\to} c_2^{\leftarrow} c_3^{\leftarrow} c_2^{\to} c_1^{\leftarrow}$$

$$l\left((((c_1^{\to})(c_2^{\leftarrow}))(((c_3^{\leftarrow})(c_2^{\to}))(c_1^{\leftarrow})))\right) = 5$$

Definition 1.6 *Let* $\mathcal{T}_{topfc}$ *be the monoidal* $\mathbf{C}$*-category that is given by the following data:*
objects: *non-associative coloured words. The unit object is the empty word, and the tensor product on the objects is defined by* $w_1 \otimes w_2 := (w_1 w_2)$.
morphisms:
generators: *The morphism spaces are generated by:*

(G1) A morphism ${}_{v,w,x}$ *and a morphism* ${}_{v,w,x}$ *for each triple* (v,w,x) *of non-empty non-associative coloured words. The sources of these morphisms are* $((vw)x)$ *and* $(v(wx))$ *and their targets are* $(v(wx))$ *and* $((vw)x)$ *respectively.*

(G2) A morphism ${}_{v,w}$ *and a morphism* ${}_{v,w}$ *for each pair* (v,w) *of non-empty non-associative coloured words. The source of these morphisms is* (vw) *and their target is* (wv).

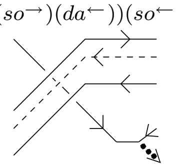
(G3) A morphism ${}_c$ *and a morphism* ${}_c$ *with source empty word and* $((c^{\overline{x}})(c^x))$ *and target* $((c^{\overline{x}})(c^x)$ *and empty word respectively for each* $c \in \mathcal{C}ol$ *and* $x \in \{\rightarrow, \leftarrow\}$.

(G4) A morphism ${}_{c_1^{x_1}, c_2^{x_2}, c_3^{x_3}}$ *and a morphism* ${}_{c_1^{x_1}, c_2^{x_2}, c_3^{x_3}}$ *with source* $(c_1^{x_1})$ *and* $((c_1^{x_1})(c_2^{x_2}))$ *and target* $((c_2^{x_2})(c_3^{x_3}))$ *and* $(c_3^{x_3})$ *respectively for each triple* $(c_1^{x_1}, c_2^{x_2}, c_3^{x_3})$ *with* $c_i \in \mathcal{C}ol$ *and* $x_i \in \{\rightarrow, \leftarrow\}$.

relations:
To every morphism of $\mathcal{T}_{topfc}$ *we can assign an oriented framed coloured 3-tangle: Proceed like in the case of* $\mathcal{T}_{tpf}$ *(see [BS]) to obtain an uncoloured, unoriented 3-tangle and then colour each edge with the colour indicated by source and/or target and orient them as indicated in the exponent of the components of the objects.*
We impose the following relation on the morphism spaces: two morphisms from u to w are equivalent if they get assigned equivalent oriented framed coloured 3-tangles.

Example 1.7 $(id_{(((so^{\rightarrow})(da^{\leftarrow}))(so^{\leftarrow}))} \otimes {}_{so^{\rightarrow}, so^{\leftarrow}, do^{\rightarrow}})\ {}_{(so^{\rightarrow}),(((so^{\rightarrow})(da^{\leftarrow}))(so^{\leftarrow}))}$

gets assigned:



Convention: In a graphical representation of a morphism, the colouring of the edges can either be read from the adjacent component(s) of their source and/or target or, if these are omitted, is indicated by the style of the line.
Example: $da^{\rightarrow}$ ——— $da^{\rightarrow}$ = - - - > - - - .

Definition 1.8 *A* coloured oriented 3-diagram *is a finite trivalent graph K (by which we understand a graph with every vertex being either univalent or trivalent or else bivalent and adjacent to a loop) equipped with the following data:*

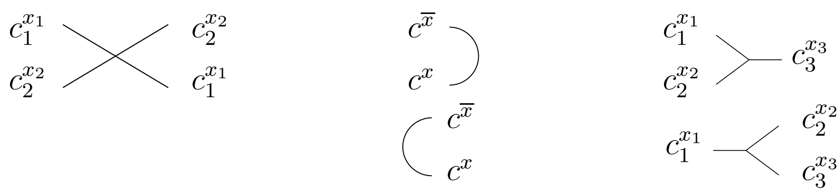
- *for every trivalent vertex x of K, a cyclic order of the edges arriving at x.*
- *a colouring of the edges of* K*, i.e. a map* $col : \{edges\ of\ K\} \rightarrow \mathcal{C}ol$.

The degree *of a 3-diagram is the number of trivalent vertices adjacent to at least one dashed edge*[1].

[1] Note that for a 3-diagram without univalent vertices adjacent to a chord, this is twice the classical degree.

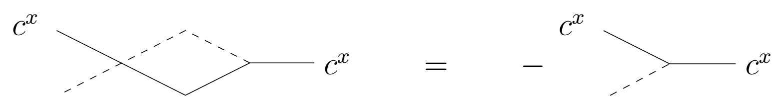
Usually, we describe the 3-diagrams by graphical representations in the plane encoding the information about the cyclic order near the trivalent vertices by arranging the adjacent edges counterclockwise.

Definition 1.9 *The category $\mathcal{D}_{3c}$ is a monoidal* $\mathbf{C}$*-category whose morphisms are linear combinations of certain graphical representations of coloured oriented 3-diagrams. It is given by the following data:*
objects: $\mathrm{Obj}(\mathcal{D}_{3c}) := \dot{\bigcup}_{n=0}^{\infty}\{c^{\rightarrow}, c^{\leftarrow} | c \in \mathcal{C}ol\}^n$. *The tensor product on* $\mathrm{Obj}(\mathcal{D}_{3c})$ *is the juxtaposition.*
morphisms:
generators: *The morphism spaces are generated by:*

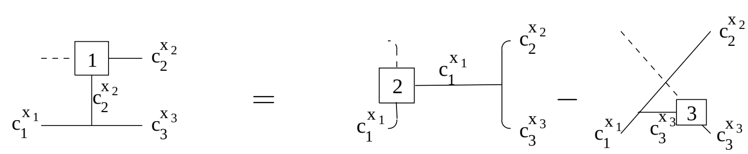


the source (resp. the target) being denoted on the left-(resp. right-)hand side from top to bottom and the edges oriented and coloured as indicated by the objects. [2]
The tensor product of two morphisms is obtained by putting the first above the second, the composition by glueing together the corresponding entries of the target of the first and the source of the second.
relations: *Of course, different graphical representations of isomorphic coloured oriented 3-diagrams are to represent the same morphism; in addition, we impose the following relations:*

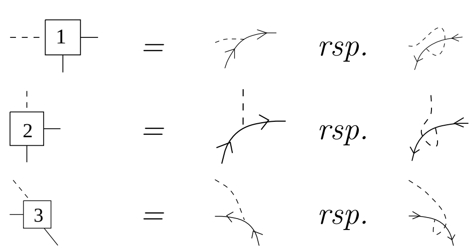
(AS)



(IHX)



with



with all possible orientations of the edges and all choices of colours $c, c_1, c_2, c_3 \in \mathcal{C}ol$.

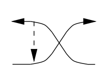
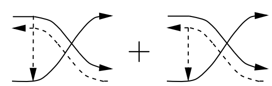
Definition 1.10 *Let $\hat{\mathcal{D}}_{3c}$ be the completion of the (graded) category $\mathcal{D}_{3c}$.*

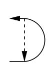
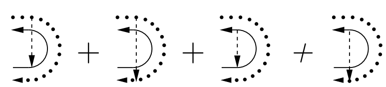
[2]The apparent 4-valent vertices are no vertices at all - they are just crossings of two edges (there is no need to say that one of them passes over the other).

For convenience of notation, we define a family of maps Δ_w. To do this, we need the following notions:

Let a *connection* in a coloured oriented 3-diagram be a sequence of solid edges such that the first and the last edge are adjacent to a univalent vertex and the head of the first and the tail of the second of any two subsequent edges is identical and the third edge arriving at this trivalent vertex is dashed.
A morphism of $\hat{\mathcal{D}}_{3c}$ may be equipped with some extra information: in each of its terms, some connections may be labelled. By this, we obtain what we will call a *labelled* morphism of $\hat{\mathcal{D}}_{3c}$.
If D is a morphism of $\hat{\mathcal{D}}_{3c}$ whose source has as k-th component a "solid", we denote by D_k the labelled morphism that is given by D and the labelling of the connection parting from the k-th component of the source in each summand.
For any non-associative coloured word w, call Δ_w the map which is defined on the space of all labelled morphisms of $\hat{\mathcal{D}}_{3c}$ as follows:
In every summand, the labelled connection is replaced by a bunch of parallel strands the uppermost of which is coloured and oriented according to the first component of w, the second according to the second component of w, and so on; then the sum over all possible ways of lifting the trivalent vertices of the original connection to the new ones is taken.

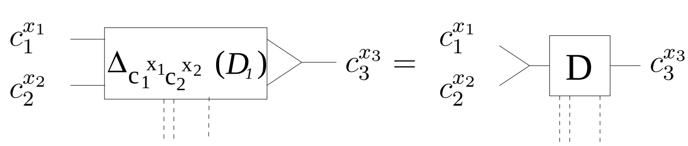
Example 1.11

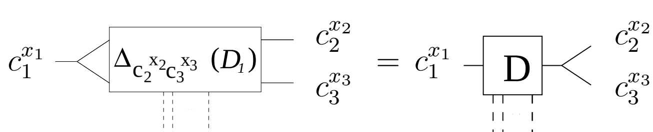
(a) $w := (so^{\rightarrow}, da^{\leftarrow})$, $D :=$  , $\Delta_w(D_1) =$ 

(b) $w := (so^{\rightarrow}, do^{\leftarrow})$, $D :=$  , $\Delta_w(D_2) =$ 

As an immediate consequence of the relation *(IHX)*, we obtain:

Lemma 1.12



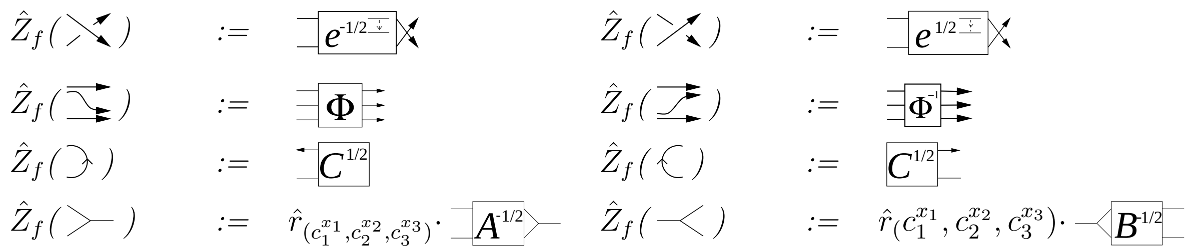


for any fitting 3-diagram D. □

Finally, we get to the definition of the universal Vassiliev-Kontsevich invariant:

Theorem and Definition 1.13 *The following assignments define a monoidal functor* $\hat{Z}_f : \mathcal{T}_{topfc} \to \hat{\mathcal{D}}_{3c}$, *the* extension of the universal Vassiliev-Kontsevich invariant *to oriented framed coloured 3-tangles:*

$\hat{Z}_f(w) := u(w)$, *where w is a non-associative coloured word.*

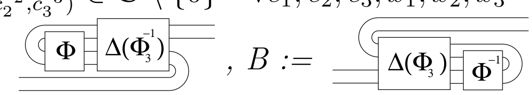


where $e^{\pm\frac{1}{2}}$ $:= \sum_{n=0}^{\infty} (\pm\frac{1}{2})^n \frac{1}{n!}$ ^{n}

Φ *is the Knizhnik-Zamolodchikov associator with arbitrary orientation on chords (for definition see [LM 3])*

$C := ($ Φ $)^{-1}$

$\hat{r}_{(c_1^{x_1},c_2^{x_2},c_3^{x_3})} \in \mathbf{C} \setminus \{0\} \quad \forall c_1, c_2, c_3, x_1, x_2, x_3$

$A :=$  .

For all the other generators (i.e. $_{u,v}$, $_{u,v}$, $_{u,v,w}$, *or* $_{u,v,w}$ *for some non empty non-associative coloured words u, v, and w, and* $_c$, $_c$ *for some colour c) the image under* $\hat{Z}_f$ *is obtained by applying* $\Delta_u, \Delta_v, \Delta_w$ *and* $\Delta_{c^{\overline{x}}}$ *to the corresponding components of* $\hat{Z}_f(\ )$, $\hat{Z}_f(\ )$, $\hat{Z}_f(\ )$, $\hat{Z}_f(\ )$, $\hat{Z}_f(\ )$, *or* $\hat{Z}_f(\ )$ *respectively, e.g.*

$$\hat{Z}_f(\ \ _{u,v}) = \Delta_v(\Delta_u(\hat{Z}_f(\ \ ))_1)_{l(u)}.$$

Remark 1.14 As Φ, A, B, C are formal power series in certain coloured 3-diagrams with degree 0-part 1, one may take their inverses and square roots by substituting x for their higher degree parts and expand the corresponding function of x in a Taylor series.

Remark 1.15 Since the number of trivalent vertices of a certain type in a 3-tangle is invariant, we have choices for $\hat{r}_{(c_1^{x_1},c_2^{x_2},c_3^{x_3})}$. The only restriction to these choices is, that $\hat{r}_{(c_1^{x_1},c_2^{x_2},c_3^{x_3})} = \hat{r}_{(c_4^{x_4},c_5^{x_5},c_6^{x_6})}$ whenever $(c_1^{x_1}, c_2^{x_2}, c_3^{x_3})$ and $(c_4^{x_4}, c_5^{x_5}, c_6^{x_6})$ differ by a permutation.

Proof of 1.13: In section 1 of [MO], Murakami and Ohtsuki define the universal Vassiliev-Kontsevich invariant for (uncoloured) oriented 3-tangles and prove that it is indeed an invariant. First we modify their definition—without destroying the invariance of the functor—in a way that keeps it well defined for unoriented 3-nets and 3-diagrams[3]: Omit the signs accounting for the orientation of the strands, and introduce the antisymmetry relation (AS) instead of the corresponding (implicit) symmetry relation. In order to obtain as degree 1-part of $\hat{Z}_f(\ ) := \hat{Z}_f(\ \ -\ \ )$ the diagram in which the double point is replaced by a chord (arriving at the support on either side like this:), we have adjusted the sign in the exponent of the image of the crossings.

[3]in the case of the Lie algebras $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_7$, and $\mathfrak{e}_8$ this is useful because it allows us to work with unoriented 3-nets (see remark 1.20)

The extension of the (uncoloured) universal Vassiliev-Kontsevich invariant to coloured 3-nets succeeds because equivalent coloured 3-nets are equivalent as uncoloured 3-nets and the colouring is preserved by the universal Vassiliev- Kontsevich invariant. □

Now we focus on the Lie algebraic part of the construction of the invariants. As we aim at skein relations related to the standard representation V of some exceptional Lie algebra $\mathfrak{g}$, we are interested in the decomposition of $V \otimes V$ into irreducible representations of $\mathfrak{g}$. In the following summary, definitions and notations are as given in [H]and used in [LCL]:

$\mathfrak{g}_2$: $V :=$ the 7-dimensional irreducible representation of $\mathfrak{g}_2$
$V \otimes V \cong \mathbf{C} \oplus W \oplus V \oplus L$
with highest weights: V: (1,0), L: (0,1) (adjoint representation), W: (2,0).

$\mathfrak{f}_4$: $V :=$ the 26-dimensional irreducible representation of $\mathfrak{f}_4$
$V \otimes V \cong \mathbf{C} \oplus V \oplus U \oplus L \oplus W$
with highest weights: V: (0,0,0,1), L: (1,0,0,0) (adjoint representation), U: (0,0,0,2), W: (0,0,1,0).

$\mathfrak{e}_6$: $V :=$ a 27-dimensional irreducible representation of $\mathfrak{e}_6$
$V \otimes V \cong U \oplus V^* \oplus W$
with highest weights: V: (0,0,0,0,0,1), V^*: (1,0,0,0,0,0), U: (0,0,0,0,0,2), W: (0,0,0,0,1,0).

$\mathfrak{e}_7$: $V :=$ the 56-dimensional irreducible representation of $\mathfrak{e}_7$
$V \otimes V \cong \mathbf{C} \oplus U \oplus L \oplus W$
with highest weights: V: (0,0,0,0,0,0,1), L: (1,0,0,0,0,0,0) (adjoint representation), U: (0,0,0,0,0,1,0), W: (0,0,0,0,0,0,2).

$\mathfrak{e}_8$: $V := L$, the adjoint representation of $\mathfrak{e}_8$
$V \otimes V \cong \mathbf{C} \oplus U \oplus X \oplus V \oplus W$
with highest weights: V: (0,0,0,0,0,0,0,1), U: (1,0,0,0,0,0,0,0), X: (0,0,0,0,0,0,0,2), W: (0,0,0,0,0,0,1,0).

Note that an occurrence of $\mathbf{C}$ in the decomposition of $V \otimes V$ implies that V is selfdual (i.e. $V \cong V^*$); so all standard representations except the one of $\mathfrak{e}_6$ are selfdual.

We now fix for each exceptional Lie algebra $\mathfrak{g}$ a set $\mathcal{S}(\mathfrak{g})$ of irreducible representations of $\mathfrak{g}$:

$\mathcal{S}(\mathfrak{g}_2) := \{\mathbf{C}, V, V^*, L, L^*\}$ $\quad$ $\mathcal{S}(\mathfrak{f}_4) := \{\mathbf{C}, V, V^*, L, L^*\}$ $\quad$ $\mathcal{S}(\mathfrak{e}_6) := \{\mathbf{C}, V, V^*, U, U^*, L\}$
$\mathcal{S}(\mathfrak{e}_7) := \{\mathbf{C}, V, V^*, L, L^*\}$ $\quad$ $\mathcal{S}(\mathfrak{e}_8) := \{\mathbf{C}, V, V^*, U, U^*\}$

Note that all these representations except $V, V^*, U,$ and U^* in the case $\mathfrak{g} = \mathfrak{e}_6$ are selfdual.
A check with the program LiE (see [LCL]) shows that for $X, Y, Z \in \mathcal{S}(\mathfrak{g})$ the dimension of $\mathrm{Hom}_{\mathfrak{g}}(X \otimes Y, Z)$ is 1 or 0; so we have a $\mathfrak{g}$-linear map $p^{\mathfrak{g}}_{X \otimes Y, Z}$ from $X \otimes Y$ to Z that is either 0 or uniquely defined up to a scalar.
This is important for proposition 1.18 and has partly motivated the choice of the $\mathcal{S}(\mathfrak{g})$.

Definition 1.16 *The family* $\{p^{\mathfrak{g}}_{X \otimes Y, Z} : X \otimes Y \to Z | X, Y \in \mathcal{S}\backslash\{\mathbf{C}\}, Z \in \mathcal{S}(\mathfrak{g})\}$ *of*

$\mathfrak{g}$-*linear maps is* consistent *if it has the following properties:*

(i) $p^{\mathfrak{g}}_{X\otimes Y,Z} \not\equiv 0$ *whenever* $\dim Hom_{\mathfrak{g}}(X\otimes Y, Z) \neq 0$ *and* $p^{\mathfrak{g}}_{L\otimes L,\mathbf{C}} = h\kappa$ *where* h *is any non-zero complex number and* κ *the Killing form,*

(ii) $p^{\mathfrak{g}}_{X\otimes X^*,\mathbf{C}}(x\otimes y) = \begin{cases} -p^{\mathfrak{g}}_{X^*\otimes X,\mathbf{C}}(y\otimes x) & (\forall x\in X, y\in X^*) \text{ if } \mathfrak{g}=\mathfrak{e}_7 \text{ and } X=V \\ p^{\mathfrak{g}}_{X^*\otimes X,\mathbf{C}}(y\otimes x) & (\forall x\in X, y\in X^*) \text{ otherwise,} \end{cases}$

(iii) For $X,Y,Z \in \mathcal{S}(\mathfrak{g})\backslash\{\mathbf{C}\}$ $p^{\mathfrak{g}}_{Y\otimes Z^*,X^*} = (id_{X^*}\otimes p^{\mathfrak{g}}_{Z\otimes Z^*,\mathbf{C}})\circ(id_{X^*}\otimes p^{\mathfrak{g}}_{X\otimes Y,Z}\otimes id_{Z^*})\circ(i^{\mathfrak{g}}_{\mathbf{C},X^*\otimes X}\otimes id_Y\otimes id_{Z^*})$ *where* $i^{\mathfrak{g}}_{X,Y\otimes Z}\in Hom(X, Y\otimes Z)$ *is the unique* $\mathfrak{g}$*-linear map with* $p^{\mathfrak{g}}_{Y\otimes Z,X}\circ i^{\mathfrak{g}}_{X,Y\otimes Z} = id_X$ *if* $p^{\mathfrak{g}}_{Y\otimes Z,X}\not\equiv 0$*, and* $i^{\mathfrak{g}}_{X,Y\otimes Z}\equiv 0$ *otherwise.*

Of course, there exist such consistent families.

For the construction of $\Psi_{\mathfrak{g}}$ we will also need the following $\mathfrak{g}$-linear maps:

$$flip_{X\otimes Y} : X\otimes Y \to Y\otimes X \quad (\forall X, Y\in\mathcal{S}(\mathfrak{g})),$$

the $\mathfrak{g}$-linear map taking $x\otimes y$ to $y\otimes x$ $\quad(\forall x\in X, y\in Y)$.

Let $\hat{h}$ be a formal parameter.

Definition 1.17 *The category* $\mathcal{C}(\mathfrak{g})$ *is the monoidal* $\mathbf{C}[[\hat{h}]]$*-category with objects* $\mathrm{Obj}(\mathcal{C}(\mathfrak{g})) := \{\mathbf{C}[[\hat{h}]]\otimes_{\mathbf{C}} U \,|\, U$ *is a tensor product over* $\mathbf{C}$ *with factors in* $\mathcal{S}(\mathfrak{g})\}$ *and with the following morphism spaces:*

$$\mathrm{Mor}_{\mathcal{C}(\mathfrak{g})}(\mathbf{C}[[\hat{h}]]\otimes_{\mathbf{C}} U_1, \mathbf{C}[[\hat{h}]]\otimes_{\mathbf{C}} U_2) := \mathbf{C}[[\hat{h}]]\otimes_{\mathbf{C}} \mathrm{Hom}_{\mathfrak{g}}(U_1, U_2).$$

The definition of $\Psi_{\mathfrak{g}}$ is contained in the following proposition whose proof will be omitted, because it just consists in checking straightforwardly that $\Psi_{\mathfrak{g}}$ respects all relations required.

Proposition 1.18 *For all consistent families* $\{p^{\mathfrak{g}}_{X\otimes Y,Z} | X,Y\in\mathcal{S}(\mathfrak{g})\backslash\mathbf{C}, Z\in\mathcal{S}(\mathfrak{g})\}$*, there exist* $k_{X,Y,Z}\in\mathbf{C}$ *for which we obtain a well defined* $\mathbf{C}[[\hat{h}]]$*-linear monoidal functor* $\Psi_{\mathfrak{g}} : \hat{\mathcal{D}}_{3c}\to\mathcal{C}(\mathfrak{g})$ *by setting*

(i) $\Psi_{\mathfrak{g}}(\textit{empty word}) := \mathbf{C}[[\hat{h}]]$

$\Psi_{\mathfrak{g}}(so^{\rightarrow}) := \mathbf{C}[[\hat{h}]]\otimes V;\qquad \Psi_{\mathfrak{g}}(da^{\leftarrow}) := \mathbf{C}[[\hat{h}]]\otimes L;$

$\Psi_{\mathfrak{g}}(so^{\leftarrow}) := \mathbf{C}[[\hat{h}]]\otimes V^*;\qquad \Psi_{\mathfrak{g}}(do^{\rightarrow}) := \begin{cases} \mathbf{C}[[\hat{h}]]\otimes U^* & \text{if } \mathfrak{g}=\mathfrak{e}_6 \text{ or } \mathfrak{e}_8 \\ \mathbf{C}[[\hat{h}]]\otimes V^* & \text{otherwise;} \end{cases}$

$\Psi_{\mathfrak{g}}(da^{\rightarrow}) := \mathbf{C}[[\hat{h}]]\otimes L;\qquad \Psi_{\mathfrak{g}}(do^{\leftarrow}) := \begin{cases} \mathbf{C}[[\hat{h}]]\otimes U & \text{if } \mathfrak{g}=\mathfrak{e}_6 \text{ or } \mathfrak{e}_8 \\ \mathbf{C}[[\hat{h}]]\otimes V & \text{otherwise.} \end{cases}$

(ii) $\Psi_{\mathfrak{g}}\left(\begin{array}{ccc} c_1^{x_1} & & c_2^{x_2} \\ & \times & \\ c_2^{x_2} & & c_1^{x_1} \end{array}\right) := \begin{cases} -1\otimes flip_{\Psi_{\mathfrak{g}}(c_1^{x_1})\otimes\Psi_{\mathfrak{g}}(c_2^{x_2})} & \text{if } \Psi_{\mathfrak{g}}(c_1^{x_1}), \Psi_{\mathfrak{g}}(c_2^{x_2})\in\{V, V^*\} \\ & \text{and } \mathfrak{g}=\mathfrak{e}_7 \\ 1\otimes flip_{\Psi_{\mathfrak{g}}(c_1^{x_1})\otimes\Psi_{\mathfrak{g}}(c_2^{x_2})} & \end{cases}$

(iii) $\Psi_{\mathfrak{g}}(\; {}^{c^{\overline{x}}}_{c^{x}} \;) := 1 \otimes p^{\mathfrak{g}}_{\Psi_{\mathfrak{g}}(c^{\overline{x}})\otimes\Psi_{\mathfrak{g}}(c^{x}),\mathbf{C}}$

$\Psi_{\mathfrak{g}}(\; {}^{c^{\overline{x}}}_{c^{x}} \;) := \dim\, \Psi_{\mathfrak{g}}(c^{x}) \otimes i^{\mathfrak{g}}_{\mathbf{C},\Psi_{\mathfrak{g}}(c^{\overline{x}})\otimes\Psi_{\mathfrak{g}}(c^{x})}$,

where $i^{\mathfrak{g}}_{\mathbf{C},X\otimes X^*} \in Hom(\mathbf{C}, X \otimes X^*)$ *is the unique* $\mathfrak{g}$*-linear map with* $p^{\mathfrak{g}}_{X\otimes X^*,\mathbf{C}} \circ i^{\mathfrak{g}}_{\mathbf{C},X\otimes X^*} = id_{\mathbf{C}}$ *if* $p^{\mathfrak{g}}_{X\otimes X^*,\mathbf{C}} \not\equiv 0$, *and* $i^{\mathfrak{g}}_{\mathbf{C},X\otimes X^*} \equiv 0$ *otherwise.*

(iv) $\Psi_{\mathfrak{g}}(\; {}^{c_1^{x_1}}_{c_2^{x_2}} \; c_3^{x_3}) := \begin{cases} 1 \otimes p^{\mathfrak{g}}_{\Psi_{\mathfrak{g}}(c_1^{x_1})\otimes\Psi_{\mathfrak{g}}(c_2^{x_2}),\Psi_{\mathfrak{g}}(c_3^{x_3})} & \textit{if } \Psi_{\mathfrak{g}}(c_i^{x_i}) \neq L (i = 1, 2, 3) \\ \hat{h} \otimes p^{\mathfrak{g}}_{\Psi_{\mathfrak{g}}(c_1^{x_1})\otimes\Psi_{\mathfrak{g}}(c_2^{x_2}),\Psi_{\mathfrak{g}}(c_3^{x_3})} & \textit{otherwise} \end{cases}$

$\Psi_{\mathfrak{g}}(c_1^{x_1} \; {}^{c_2^{x_2}}_{c_3^{x_3}} \;) := \begin{cases} k^{\mathfrak{g}}_{\Psi_{\mathfrak{g}}(c_1^{x_1}),\Psi_{\mathfrak{g}}(c_2^{x_2}),\Psi_{\mathfrak{g}}(c_3^{x_3})} \otimes i^{\mathfrak{g}}_{\Psi_{\mathfrak{g}}(c_1^{x_1}),\Psi_{\mathfrak{g}}(c_2^{x_2})\otimes\Psi_{\mathfrak{g}}(c_3^{x_3})} \\ \quad \textit{if } \Psi_{\mathfrak{g}}(c_i^{x_i}) \neq L \; (i = 1, 2, 3) \\ k^{\mathfrak{g}}_{\Psi_{\mathfrak{g}}(c_1^{x_1}),\Psi_{\mathfrak{g}}(c_2^{x_2}),\Psi_{\mathfrak{g}}(c_3^{x_3})} \hat{h} \otimes i^{\mathfrak{g}}_{\Psi_{\mathfrak{g}}(c_1^{x_1}),\Psi_{\mathfrak{g}}(c_2^{x_2})\otimes\Psi_{\mathfrak{g}}(c_3^{x_3})} \\ \quad \textit{otherwise,} \end{cases}$

where $i^{\mathfrak{g}}_{X,Y\otimes Z} \in Hom(X, Y \otimes Z)$ *is the unique* $\mathfrak{g}$*-linear map with* $p^{\mathfrak{g}}_{Y\otimes Z,X} \circ i^{\mathfrak{g}}_{X,Y\otimes Z} = id_X$ *if* $p^{\mathfrak{g}}_{Y\otimes Z,X} \not\equiv 0$, *and* $i^{\mathfrak{g}}_{X,Y\otimes Z} \equiv 0$ *otherwise.*

Remark 1.19 The factors $k^{\mathfrak{g}}_{X,Y,Z}$ in (iv) depend on the maps $p^{\mathfrak{g}}_{X\otimes Y,Z} : X \otimes Y \to Z$. The value of $k^{\mathfrak{g}}_{X,Y,Z}$ for a fixed map $p^{\mathfrak{g}}_{X\otimes Y,Z}$ can be found by solving the equation $(p^{\mathfrak{g}}_{Y^*\otimes Y,\mathbf{C}} \otimes id_Z) = p^{\mathfrak{g}}_{Y^*\otimes X,Z} \circ (id_{Y^*} \otimes i^{\mathfrak{g}}_{X,Y\otimes Z})$ representing the fact that $\Psi_{\mathfrak{g}}(\quad) = \Psi_{\mathfrak{g}}(\quad)$ must hold.
The factor $\dim\, \Psi_{\mathfrak{g}}(c^x)$ in (iii) has been chosen to assure $\Psi_{\mathfrak{g}}(\quad) = \Psi_{\mathfrak{g}}(\quad)$; it is independent of the maps $p^{\mathfrak{g}}_{c^{\overline{x}}\otimes c^{x},\mathbf{C}}$.
The formal parameter $\hat{h}$ has been introduced to assure the existence of $\Psi_{\mathfrak{g}}(D)$ for every morphism D of $\hat{\mathcal{D}}_{3c}$: Due to $\hat{h}$, the morphism spaces of $\mathcal{C}(\mathfrak{g})$ can be regarded as graded spaces in the obvious way. This makes $\Psi_{\mathfrak{g}}$ a grade preserving functor that is well defined in every degree and hence on the whole of $\hat{\mathcal{D}}_{3c}$.

Remark 1.20 For $\mathfrak{g} = \mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_7$, and $\mathfrak{e}_8$ the selfduality of all the representations in $\mathcal{S}(\mathfrak{g})$ allows us to identify V, U, and L with V^*,U^*, and L^* respectively (by fixed isomorphisms). Therefore, for any $D \in \mathcal{D}_{3c}$, $\Psi_{\mathfrak{g}}(D)$ does not depend on the orientation of the edges and $I_{(\mathfrak{g}_2,V)}$, $I_{(\mathfrak{f}_4,V)}$,$I_{(\mathfrak{e}_7,V)}$, and $I_{(\mathfrak{e}_8,V)}$ are well-defined on unoriented 3-tangles.

2 Eigenvalue tables

What we have achieved so far, is the construction of an invariant for coloured 3-tangles—$I_{(\mathfrak{g},V)} := \Psi \circ \hat{Z}_f$—for any exceptional Lie algebra $\mathfrak{g}$ and its standard representation V. Unfortunately, we cannot evaluate these invariants directly because the expression known for the associator Φ is not explicit enough to allow concrete computations. But we will derive skein relations for each invariant coming from an exceptional simple Lie algebra and its standard representation V. These skein relations imply recursive rules by which we can reduce the problem of computing the invariants to finding their values for planar coloured 3-tangles.

The idea is to cut out a small neighbourhood of a crossing and insert something else without changing the value of the invariant. The substitute for the crossing has to be a linear combination of small, simple coloured 3-tangles with four univalent vertices. Obvious candidates for such are the inverse crossing, , and, in the case of unoriented 3-nets, ; as their values under $I_{(\mathfrak{g},V)}$ will prove to be linearly independent in the endomorphism space $\mathrm{End}_{\mathfrak{g}} V \otimes V$, we include with different colourings of the edge in the middle into our considerations.

For each V considered, $V \otimes V$ decomposes into several pairwise non-isomorphic irreducible subrepresentations. Hence according to the lemma of Schur, the irreducible subrepresentations are eigenspaces for each $\mathfrak{g}$-endomorphism ϕ of $V \otimes V$, i.e. ϕ is determined by the corresponding eigenvalues. These we must know to establish the skein relations.

The eigenvalues of $I_{(\mathfrak{g},V)}(\ )$ and $I_{(\mathfrak{g},V)}(\ )$ are the products of the corresponding eigenvalues of $\Psi(e^{\mp\frac{1}{2}}\ )$ and $\Psi(\ )$. Therefore, we need to know the eigenvalues of $\Psi(\ )$. To ascertain them we need to know the Lie algebra and its representation explicitly. As the corresponding tables have not been available, here a sketch of the way how we have computed them (not mentioning the various obstacles you meet there...):

From the Dynkin diagram characterizing the Lie algebra, the set of roots of the Lie algebra can be derived; this is done in [SK]. Then one can successively determine the Lie bracket of any two basis vectors of the Lie algebra. With the help of the program LiE (see [LCL]), one can find out the heighest weight of the standard representation V of the Lie algebra in question. Letting operate the Weyl group on this heighest weight, one gets the convex hull of the weight lattice of the representation, hence all weights. With the dimension of V in mind (program LiE !), one can easily write down a basis of the weight spaces of V, whereas the construction of a table for the operation of the Lie algebra on V is a little trickier. The last thing you need for the description of the map $\Psi(\ )$ is the Killing form.

In view of the dimensions of the objects involved, it is clear that these computations are to be made by computer. Our programs and the various tables they produce are available at `www.math-stat.unibe.ch/~bergerab`.

The map $I_{(\mathfrak{g},V)}(\ )$ is the identity on $V \otimes V$, hence its only eigenvalue is 1.

The eigenvalue of $I_{(\mathfrak{g},V)}(\ )$ on $\mathbf{C}$ (if $\mathbf{C}$ does appear in the decomposition of $V \otimes V$) equals $I_{(\mathfrak{g},V)}(\ )$, which is given by a formula of Rosso and Jones (see [RJ]). On all other subrepresentations, $I_{(\mathfrak{g},V)}(\ )$ is $\equiv 0$. Let c be the eigenvalue of $(\ \boxed{\Phi}\ )^{-1}$ on $\mathbf{C}$. Then $I_{(\mathfrak{g},V)}(\ := \dim V \cdot c$.

The value of the invariant $I_{(\mathfrak{g},V)}$ on a tangle of the form is a multiple of the projection onto the representation corresponding to the colouring of the edge in the middle composed with its re-imbedding. As the number of trivalent vertices of the kind appearing in this tangle is constant under isotopy, the factor (i.e. the eigenvalue on the corresponding representation) may be chosen freely.

The following tables summarize the results of our computations for all exceptional Lie algebras:

For the Lie algebra $\mathfrak{g}_2$ and its 7-dimensional standard representation V ($q = e^{-\frac{\hbar^2}{24h}}$; heighest weights of the representations: V (1,0), L (0,1), W (2,0)):

Eigenvalue on		$I_{(\mathfrak{g}_2,V)}(\ )$	$I_{(\mathfrak{g}_2,V)}(\ )$	$I_{(\mathfrak{g}_2,V)}(\ )$	$I_{(\mathfrak{g}_2,V)}(\ )$	$I_{(\mathfrak{g}_2,V)}(\ )$
S^2V	$\mathbf{C}$	q^{-6}	q^{6}	1	$7c$	0
	W	q	q^{-1}	1	0	0
$\wedge^2V$	V	$-q^{-3}$	$-q^{3}$	1	0	r
	L	-1	-1	1	0	0

For the Lie algebra $\mathfrak{f}_4$ and its 26-dimensional standard representation V ($q = e^{-\frac{\hbar^2}{36h}}, I_{\mathfrak{f}_4} := I_{(\mathfrak{f}_4,V)}$; heighest weights of the representations: V (0,0,0,1), L (1,0,0,0), U (0,0,0,2), W (0,0,1,0)):

Eigenvalue on		$I_{\mathfrak{f}_4}(\ )$	$I_{\mathfrak{f}_4}(\ )$	$I_{\mathfrak{f}_4}(\ )$	$I_{\mathfrak{f}_4}(\ )$	$I_{\mathfrak{f}_4}(\ )$	$I_{\mathfrak{f}_4}(\ )$
S^2V	$\mathbf{C}$	q^{-12}	q^{12}	1	$26c$	0	0
	V	q^{-6}	q^{6}	1	0	r	0
	U	q	q^{-1}	1	0	0	0
$\wedge^2V$	L	$-q^{-3}$	$-q^{3}$	1	0	0	s
	W	-1	-1	1	0	0	0

For the Lie algebra $\mathfrak{e}_6$ and its 27-dimensional standard representation V ($q = e^{-\frac{\hbar^2}{72h}}$; heighest weights of the representations: V (0,0,0,0,0,1), U (0,0,0,0,0,2), V^* (1,0,0,0,0,0), W (0,0,0,0,1,0)):

Eigenvalue on		$I_{(\mathfrak{e}_6,V)}(\ )$	$I_{(\mathfrak{e}_6,V)}(\ )$	$I_{(\mathfrak{e}_6,V)}(\ )$	$I_{(\mathfrak{e}_6,V)}(\ )$	$I_{(\mathfrak{e}_6,V)}(\ )$
S^2V	U	q^{2}	q^{-2}	1	0	s
	V^*	q^{-13}	q^{13}	1	r	0
$\wedge^2V$	W	$-q^{-1}$	$-q$	1	0	0

For the Lie algebra $\mathfrak{e}_7$ and its 56-dimensional standard representation V ($q = e^{-\frac{\hbar^2}{144h}}$; heighest weights of the representations: V (0,0,0,0,0,0,1), L (1,0,0,0,0,0,0), U (0,0,0,0,0,1,0), W (0,0,0,0,0,0,2)):

Eigenvalue on		$I_{(\mathfrak{e}_7,V)}(\ )$	$I_{(\mathfrak{e}_7,V)}(\ )$	$I_{(\mathfrak{e}_7,V)}(\ )$	$I_{(\mathfrak{e}_7,V)}(\ )$	$I_{(\mathfrak{e}_7,V)}(\ )$
$\wedge^2V$	$\mathbf{C}$	q^{-57}	q^{57}	1	$56c$	0
	U	q^{-1}	q	1	0	0
S^2V	L	$-q^{-21}$	$-q^{21}$	1	0	s
	W	$-q^{3}$	$-q^{-3}$	1	0	0

For the Lie algebra $\mathfrak{e}_8$ and its adjoint representation L as standard representa-

tion ($q = e^{-\frac{\hat{h}^2}{30h}}$, $I_{\mathfrak{e}_8} := I_{(\mathfrak{e}_8,L)}$; heighest weights of the representations: X (0,0,0,0,0,0,0,2), L (0,0,0,0,0,0,0,1), U (1,0,0,0,0,0,0,0), W (0,0,0,0,0,0,1,0)):

Eigen-value on		$I_{\mathfrak{e}_8}(\ )$	$I_{\mathfrak{e}_8}(\ )$	$I_{\mathfrak{e}_8}(\ )$	$I_{\mathfrak{e}_8}(\ )$	$I_{\mathfrak{e}_8}(\ )$	$I_{\mathfrak{e}_8}(\ )$
S^2L	**C**	q^{30}	q^{-30}	1	$248c$	0	0
	U	q^{-6}	q^{6}	1	0	0	s
	X	q	q^{-1}	1	0	0	0
$\wedge^2L$	L	$-q^{-15}$	$-q^{15}$	1	0	r	0
	W	-1	-1	1	0	0	0

3 Skein relations

Now it is easy to establish skein relations for the invariants coming from the various exceptional Lie algebras: we just have to find out how to express the eigenvalue vector corresponding to the positive crossing as a linear combination of the eigenvalue vectors corresponding to the other elementary tangles. But to be sure to end up with planar coloured oriented 3-nets, skein relations involving only one crossing are needed. In the case of the Lie algebra $\mathfrak{e}_6$, we have solved this problem by introducing . In the remaining cases, we could of course have proceeded by using further colours, too; but we have found rather sophisticated methods to avoid this. We describe them in detail for the Lie algebras $\mathfrak{g}_2$ and $\mathfrak{f}_4$; the cases of $\mathfrak{e}_7$ and $\mathfrak{e}_8$ can be treated in the same way as the ones of $\mathfrak{g}_2$ and $\mathfrak{f}_4$, respectively.

For the Lie algebra $\mathfrak{g}_2$ and its 7-dimensional irred. representation V:

$$I_{(\mathfrak{g}_2,V)}(\ ) = \alpha I_{(\mathfrak{g}_2,V)}(\ ) + \beta I_{(\mathfrak{g}_2,V)}(\ ) + \gamma I_{(\mathfrak{g}_2,V)}(\ ) + \delta I_{(\mathfrak{g}_2,V)}(\ )$$

where $\alpha := q$, $\beta := q-1$, $\gamma := \frac{1}{7c}(-q^7 + q^{-6} - q + 1)$, $\delta := \frac{1}{r}(q^4 - q^{-3} - q + 1)$.

Since $I_{(\mathfrak{g}_2,V)}$ is a monoidal functor and invariant under ambient isotopy, we can deduce a new skein relation as follows:

$$\begin{aligned} I_{(\mathfrak{g}_2,V)}(\ ) &= I_{(\mathfrak{g}_2,V)}(\ ) \\ &= \alpha I_{(\mathfrak{g}_2,V)}(\ ) + \beta I_{(\mathfrak{g}_2,V)}(\ ) + \gamma I_{(\mathfrak{g}_2,V)}(\ ) + \delta I_{(\mathfrak{g}_2,V)}(\ ) \\ &= \alpha I_{(\mathfrak{g}_2,V)}(\ ) + \beta I_{(\mathfrak{g}_2,V)}(\ ) + \gamma I_{(\mathfrak{g}_2,V)}(\ ) + \delta I_{(\mathfrak{g}_2,V)}(\ ). \end{aligned}$$

Combining these two skein relations, we obtain:

$$I_{(\mathfrak{g}_2,V)}(\ ) = \lambda I_{(\mathfrak{g}_2,V)}(\ ) + \mu I_{(\mathfrak{g}_2,V)}(\ ) + \rho I_{(\mathfrak{g}_2,V)}(\ ) + \sigma I_{(\mathfrak{g}_2,V)}(\ )$$

where $\lambda := \frac{q^2}{q+1}$, $\mu := \frac{1}{q(q+1)}$, $\rho := -\frac{q^6+q^5+q^4+q^2+q+1}{q^3(q+1)r}$, $\sigma := q\rho$.

For the Lie algebra $\mathfrak{f}_4$ and its 26-dimensional irred. representation V:

$$I_{(\mathfrak{f}_4,V)}(\ ) = \alpha I_{(\mathfrak{f}_4,V)}(\ ) + \beta I_{(\mathfrak{f}_4,V)}(\ ) + \gamma I_{(\mathfrak{f}_4,V)}(\ ) + \delta I_{(\mathfrak{f}_4,V)}(\ ) + \epsilon I_{(\mathfrak{f}_4,V)}(\ ) \tag{1}$$

where $\alpha := q$, $\beta := q-1$, $\gamma := -\frac{q^9-q^8-q^5+q^4+q-1}{(q^6-q^3+1)q}$, $\delta := -\frac{q^{13}+q^7-q^6-1}{rq^6}$, $\epsilon := \frac{q^7-q^4+q^3-1}{sq^3}$.

To establish a skein relation involving only one crossing, the tangles given in the table in section 2 are not enough. But we can also derive the eigenvalues $x_{\mathbf{C}}, x_V, x_L, x_U$, and x_W for $I_{(\mathfrak{f}_4,V)}(\ )$ on $\mathbf{C}, V, L, U$, and W respectively (and as a byproduct the eigenvalues $y_{\mathbf{C}}, y_V, y_L, y_U$, and y_W for $I_{(\mathfrak{f}_4,V)}(\ ))$. To do this, we will apply the following strategy: first we compute $x_{\mathbf{C}}$ and $y_{\mathbf{C}}$, then we use (1) to establish relations between x_V, y_V, x_L, and y_L, and finally by examining an ansatz for a skein relation we can set up a system of equations whose solution leads to all the eigenvalues for $I_{(\mathfrak{f}_4,V)}(\ )$ and $I_{(\mathfrak{f}_4,V)}(\ )$.

(i) Eigenvalues on $\mathbf{C}$:
$x_{\mathbf{C}} = r$ (because by proceeding as in the proof of lemma 5.5 in [BS], we get $I_{(\mathfrak{f}_4,V)}(\ ) = I_{(\mathfrak{f}_4,V)}(\ ) = rI_{(\mathfrak{f}_4,V)}(\ ))$.

$y_{\mathbf{C}} = c_L^{-2}c^2 s$ with $c_L = I_{(\mathfrak{f}_4,L)}(\ )$, because $I_{(\mathfrak{f}_4,V)}(\ ) = I_{(\mathfrak{f}_4,V)}(\ ) = uI_{(\mathfrak{f}_4,V)}(\ )$ and u can be found by

$$I_{(\mathfrak{f}_4,V)}(\ ) = \hat{r}^2_{(so,da,so)}\Psi_{\mathfrak{f}_4}(\ \boxed{\Delta(C)}\ \boxed{C^{-1}}\ \boxed{C^{-1}}\ ) = \hat{r}^2_{(so,da,so)}\Psi_{\mathfrak{f}_4}(\ \boxed{C}\ \boxed{C^{-1}}\ \boxed{C^{-1}}\ ) = \hat{r}^2_{(so,da,so)}c_L^{-1}\Psi_{\mathfrak{f}_4}(\ )$$

and

$$s\Psi_{\mathfrak{f}_4}(\ ) = I_{(\mathfrak{f}_4,V)}(\ ) = \hat{r}^2_{(so,da,so)}\Psi_{\mathfrak{f}_4}(\ \boxed{C}\ \boxed{C^{-1}}\ \boxed{C^{-1}}\ ) = \hat{r}^2_{(so,da,so)}c_L c^{-2}\Psi_{\mathfrak{f}_4}(\ ).$$

(ii) Eigenvalues on V and L:
The following relations between the eigenvalues x_V, y_V, x_L and y_L hold:

$$y_V = \frac{1}{\epsilon}(q^6 - \alpha q^{-6} - \gamma + \delta x_V)$$

$$y_L = \frac{1}{\epsilon}(-q^{-3} + \alpha q^3 - \gamma - \delta x_L).$$

These relations have been derived by applying the skein relation 1 to the crossing in resp .

(iii) Note that for the skein relations

$$I_{(\mathfrak{f}_4,V)}(\ ) + f_1 I_{(\mathfrak{f}_4,V)}(\ ) = f_2 I_{(\mathfrak{f}_4,V)}(\ ) + f_3 I_{(\mathfrak{f}_4,V)}(\ ) + f_4 I_{(\mathfrak{f}_4,V)}(\ ) + f_5 (I_{(\mathfrak{f}_4,V)}\ ) \tag{2}$$

and

$$I_{(\mathfrak{f}_4,V)}(\ ) + g_1 I_{(\mathfrak{f}_4,V)}(\ ) = g_2 I_{(\mathfrak{f}_4,V)}(\ ) + g_3 I_{(\mathfrak{f}_4,V)}(\ ) + g_4 I_{(\mathfrak{f}_4,V)}(\ ) + g_5 I_{(\mathfrak{f}_4,V)}(\ ) \tag{3}$$

turning by 90° implies that $f_1 = 1, f_2 = f_3, f_4 = f_5, g_1 = 1, g_2 = g_3$, and $g_4 = g_5$. By writing down (2) and (3) for the eigenspaces $\mathbf{C}, V$, and L and using the

information found in (i) and (ii), we get the following system of equations that can be solved by maple:

$$
\begin{array}{llcl}
\mathbf{C}: & q^{-12}+q^{12} & = & f_2((1+26c)+f_4 r \\
 & q^{-12}+q^{12} & = & g_2(1+26c)+g_4 c_L^{-2} c^2 s \\
V: & q^{-6}+q^{6} & = & f_2(1+0)+f_4(r+x_V) \\
 & q^{-6}+q^{6} & = & g_2(1+0)+g_4(0+\frac{1}{\epsilon}(-q^{-3}+\alpha q^3-\gamma+\delta x_V) \\
L: & -q^{-3}-q^{3} & = & f_2(1+0)+f_4(0+x_L) \\
 & -q^{-3}-q^{3} & = & g_2(1+0)+g_4(s+\frac{1}{\epsilon}(-q^{-3}+\alpha q^3-\gamma-\delta x_L)
\end{array}
$$

Knowing f_2 and f_4, we can use (2) to get all the eigenvalues for $I_{(\mathfrak{f}_4,V)}(\;)$, and once they are known, one can establish a skein relation of the following form:

$$I_{(\mathfrak{f}_4,V)}(\;)=\lambda I_{(\mathfrak{f}_4,V)}(\;)+\mu I_{(\mathfrak{f}_4,V)}(\;)+\nu I_{(\mathfrak{f}_4,V)}(\;)+\rho I_{(\mathfrak{f}_4,V)}(\;)+\sigma I_{(\mathfrak{f}_4,V)}(\;)$$

For the Lie algebra $\mathfrak{e}_6$ and its 27-dimensional irred. representation V:

$$I_{(\mathfrak{e}_6,V)}(\;)=\alpha I_{(\mathfrak{e}_6,V)}(\;)+\beta I_{(\mathfrak{e}_6,V)}(\;)+\gamma I_{(\mathfrak{f}_4,V)}(\;)$$

where $\alpha := q, \beta := \frac{q^3-1}{q}, \gamma := -\frac{q^{27}+q^{15}-q^{12}-1}{rq^{13}}$.

$$I_{(\mathfrak{e}_6,V)}(\;)=\lambda I_{(\mathfrak{e}_6,V)}(\;)+\mu I_{(\mathfrak{e}_6,V)}(\;)+\rho I_{(\mathfrak{e}_6,V)}(\;)$$

where $\lambda := -q^{-1}$, $\mu := \frac{q^{-1}+q^{-13}}{r}$, $\rho := -\frac{q^{-1}+q^2}{s}$.

For the Lie algebra $\mathfrak{e}_7$ and its 56-dimensional irred. representation V:

$$I_{(\mathfrak{e}_7,V)}(\;)=\alpha I_{(\mathfrak{e}_7,V)}(\;)+\beta I_{(\mathfrak{e}_7,V)}(\;)+\gamma I_{(\mathfrak{e}_7,V)}(\;)+\delta I_{(\mathfrak{e}_7,V)}(\;)$$

where $\alpha := q^2$, $\beta := \frac{q^4-1}{q}$, $\delta := -\frac{1}{s}\,(q^{23}+q^3-q^{-1}-q^{-21})$,

$\gamma := \frac{q^{60}-q^{58}+q^{32}-q^{30}+1-q^{-2}+q^{-28}-q^{-30}}{(q^{16}-q^{14}+q^{12}-q^{10}+q^8-q^6+q^4-q^2+1)(q^2+1)(q^{10}+1)}$.

$$I_{(\mathfrak{e}_7,V)}(\;)=\lambda I_{(\mathfrak{e}_7,V)}(\;)+\mu I_{(\mathfrak{e}_7,V)}(\;)+\rho I_{(\mathfrak{e}_7,V)}(\;)+\sigma I_{(\mathfrak{e}_7,V)}(\;)$$

where $\lambda := -\frac{q^{88}+q^{60}+q^{57}+q^{55}+q^{47}+q^{45}+q^{39}+q^{37}+q^{29}+q^{28}+q^{27}+1}{(q^{22}-q^{16}+q^{12}+q^{10}-q^6+1)(q^2+1)^2(q^4-q^2+1)q^{28}}$,

$\mu := -\frac{q^{88}+q^{61}+q^{60}+q^{59}+q^{51}+q^{49}+q^{43}+q^{41}+q^{33}+q^{31}+q^{28}+1}{(q^{22}-q^{16}+q^{12}+q^{10}-q^6+1)(q^2+1)^2(q^4-q^2+1)q^{30}}$,

$\rho := -\frac{(q^{40}+q^{36}+q^{32}+q^{28}+q^{24}+2q^{20}+q^{16}+q^{12}+q^8+q^4+1)}{q^{21}s}$,

$\sigma := \frac{(q^4+1)(q^4-q^2+1)(q^{32}+q^{30}-q^{26}+q^{22}+q^{20}+q^{12}+q^{10}-q^6+q^2+1)}{q^{19}s}$.

For the Lie algebra $\mathfrak{e}_8$ and its adjoint representation L:

$$I_{(\mathfrak{e}_8,L)}(\;)=\alpha I_{(\mathfrak{e}_8,L)}(\;)+\beta I_{(\mathfrak{e}_8,L)}(\;)+\gamma I_{(\mathfrak{e}_8,L)}(\;)+\delta I_{(\mathfrak{e}_8,L)}(\;)+\epsilon I_{(\mathfrak{e}_8,L)}(\;)$$

where $\alpha := q$, $\beta := q-1$, $\gamma := \frac{1}{248c}\,(q^{30}-q+1-q^{-29})$,

$\delta := \frac{1}{r}\,(q^{16}-q+1-q^{-15})$, $\epsilon := -\frac{1}{s}\,(q^7+q-1-q^{-6})$.

$$
\begin{aligned}
I_{(\mathfrak{e}_8,L)}(\ ) &= \lambda I_{(\mathfrak{e}_8,L)}(\ ) + \mu I_{(\mathfrak{e}_8,L)}(\ ) + \nu I_{(\mathfrak{e}_8,L)}(\ ) \\
&\quad + \rho I_{(\mathfrak{e}_8,L)}(\ ) + \sigma I_{(\mathfrak{e}_8,L)}(\ ) + \tau I_{(\mathfrak{e}_8,L)}(\ )
\end{aligned}
$$

where $\lambda := q\mu$, $\mu := -\frac{1}{248c(q+1)}\sum_{k:=1}^{29}(q^k + q^{-k}) - \frac{q}{c(q+1)}$,

$$\nu := \tfrac{1}{r}\,(q^{16} - q + 1 - q^{-15}),\ \rho := q\nu,\ \sigma := -\tfrac{1}{s}\,(q^7 + q - 1 - q^{-6}),\ \tau := q\sigma.$$

Remember that $248c = I_{(\mathfrak{e}_8,L)}(\ )$.

4 Reducing planar coloured 3-nets

The idea how to reduce planar coloured 3-nets is essentially the same as for the skein relation: cut out a small part of the 3-net and insert something else without changing the value of the invariant. In this case, we are to cut out a small neighbourhood of a mesh (for definition see below), and the substitute has to be a linear combination of planar coloured 3-nets without bounded meshes and with the appropriate number of univalent vertices.

As $I_{(\mathfrak{g},V)}$ is a monoidal functor, it is enough to consider connected coloured 3-nets; furthermore, each planar coloured 3-net is by definition equivalent to one contained in $\mathbf{R}^2 \times \{0\}$ with only upward pointing vectors assigned, and so we can concentrate on this type of 3-nets.

Definition 4.1 *Let N be a planar coloured 3-net. A* mesh *of N is the closure of a connected component of $(\mathbf{R}^2 \times \{0\})\backslash N$. A $n-$mesh is a mesh with n trivalent vertices in the boundary.*

A planar coloured 3-net occuring as a result of the reduction of a knot by means of the skein relation is closed; thus, closed, planar coloured 3-nets are of special interest to us. About these, we have the following lemma:

Lemma 4.2 *([BS] section 5) Let N be a non-empty closed, connected, planar coloured 3-net. Then N has at least one simply connected n-mesh with $n \leq 5$.*

Unfortunately, we are not able to resolve every possible mesh; there may even be meshes that cannot be replaced by a linear combination of planar coloured 3-nets without bounded meshes (since in some cases, the dimension of the homomorphism space into which the mesh is mapped is bigger than the number of suitable 3-nets to replace it). Nevertheless, our results make it possible to calculate the values of these invariants on many knots.

In the sequel, we will report the state of affairs for each case of an exceptional simple Lie algebra. Note that the value of $I_{(\mathfrak{g},V)}$ is 0 on $\begin{matrix} c_1^{x_1} \\ c_2^{x_2} \end{matrix} \succ\!\!- \, c_3^{x_3}$ if $\Psi_{\mathfrak{g}}(c_3^{x_3})$ does not occur as direct summand in $\Psi_{\mathfrak{g}}(c_1^{x_1}) \otimes \Psi_{\mathfrak{g}}(c_2^{x_2})$. Thus coloured 3-nets containing such "branchings" are irrelevant for our purpose. We will indicate for each Lie algebra the relevant "branching types"; with this information, it is easy to write down a complete list of possible (relevant) n-meshes for any n. Exemplarily, we show in the case of $\mathfrak{e}_7$ how to derive the resolutions of all possible n-meshes for $n \leq 3$ and of a few 4-meshes.

For the Lie algebra $\mathfrak{g}_2$ and its 7-dimensional irred. representation V:
In the skein relations, only solid edges appear, and in [BS], we have showed how to resolve every solid n-mesh for $n \leq 5$. Hence we can evaluate $I_{(\mathfrak{g}_2,V)}$ on every knot.

For the Lie algebra $\mathfrak{f}_4$ and its 26-dimensional irred. representation V:
Relevant branching types: , , and .
We know how to resolve any n-mesh for $n \leq 3$ and some 4-meshes.

For the Lie algebra $\mathfrak{e}_6$ and its 27-dimensional irred. representation V:
Relevant branching types: , , , , , and .
Because of the orientations, only n-meshes with n pair can occur.
We know how to resolve any 0- and 2-mesh and some 4-meshes.

For the Lie algebra $\mathfrak{e}_7$ and its 56-dimensional irred. representation V:
Relevant branching types: and .
Let us denote by u the eigenvalue of $I_{(\mathfrak{e}_7,V)}(\ )$ on L, by f the eigenvalue of $I_{(\mathfrak{e}_7,V)}(\ )$ on $\mathbf{C}$, and by l the eigenvalue of $I_{(\mathfrak{e}_7,V)}(\ )$ on L.
0-meshes: and .
$I_{(\mathfrak{e}_7,V)}(\ )$ and $I_{(\mathfrak{e}_7,V)}(\ )$ can be computed by the formula of Rosso and Jones (see [RJ]).
1-meshes: and .
$I_{(\mathfrak{e}_7,V)}(\ )$ and $I_{(\mathfrak{e}_7,V)}(\ )$ are elements of $\mathrm{Hom}_{\mathfrak{e}_7}(L,\mathbf{C})$ and therefore $\equiv 0$.
2-meshes: , , and .
$I_{(\mathfrak{e}_7,V)}(\ ) = I_{(\mathfrak{e}_7,V)}(\ ) \circ I_{(\mathfrak{e}_7,V)}(\ )$
$=$ (eigenvalue of $I_{(\mathfrak{e}_7,V)}(\ )$ on L)(eigenvalue of $I_{(\mathfrak{e}_7,V)}(\ )$ on L)$I_{(\mathfrak{e}_7,V)}(\ )$
$=$ (eigenvalue of $I_{(\mathfrak{e}_7,V)}(\ )$ on L)$I_{(\mathfrak{e}_7,V)}(\ ) = sI_{(\mathfrak{e}_7,V)}(\ )$.
Analogously: $I_{(\mathfrak{e}_7,V)}(\ ) = uI_{(\mathfrak{e}_7,V)}(\ )$.
By means of the skein relation, we obtain:
$I_{(\mathfrak{e}_7,V)}(\ ) = \frac{1}{\sigma}(I_{(\mathfrak{e}_7,V)}(\ ) - \lambda I_{(\mathfrak{e}_7,V)}(\ ) - \mu c I_{(\mathfrak{e}_7,V)}(\ ) - \rho I_{(\mathfrak{e}_7,V)}(\ ))$
$= \frac{1}{\sigma}(f - \lambda + \mu c) I_{(\mathfrak{e}_7,V)}(\ )$.
Therefore, we have: $I_{(\mathfrak{e}_7,V)}(\ ) = I_{(\mathfrak{e}_7,V)}(\ ) = \frac{1}{\sigma}(f - \lambda + \mu c) I_{(\mathfrak{e}_7,V)}(\ )$.
3-meshes: , , , and
The skein relation yields: $I_{(\mathfrak{e}_7,V)}(\ ) =$
$\frac{1}{\sigma}(I_{(\mathfrak{e}_7,V)}(\ ) - \lambda I_{(\mathfrak{e}_7,V)}(\ ) - \mu I_{(\mathfrak{e}_7,V)}(\ ) - \rho I_{(\mathfrak{e}_7,V)}(\ ))$
$= \frac{1}{\sigma}(l - \lambda - \rho s) I_{(\mathfrak{e}_7,V)}(\ )$.
The cases of and are treated simultaneously. By setting $I_{(\mathfrak{e}_7,V)}(\ ) =: tI_{(\mathfrak{e}_7,V)}(\ )$ and $I_{(\mathfrak{e}_7,V)}(\ ) =: kI_{(\mathfrak{e}_7,V)}(\ )$, we obtain the two equations (for the second, we use the resolution of the 4-mesh given below):
$ksI_{(\mathfrak{e}_7,V)}(\ ) = kI_{(\mathfrak{e}_7,V)}(\ ) = I_{(\mathfrak{e}_7,V)}(\ ) = tI_{(\mathfrak{e}_7,V)}(\ )$
$= tuI_{(\mathfrak{e}_7,V)}(\ ) \Rightarrow ks = tu$
$gktI_{(\mathfrak{e}_7,V)}(\ ) = ktI_{(\mathfrak{e}_7,V)}(\ ) = tI_{(\mathfrak{e}_7,V)}(\ ) = I_{(\mathfrak{e}_7,V)}(\ )$
$= x_1 I_{(\mathfrak{e}_7,V)}(\ ) + x_2 I_{(\mathfrak{e}_7,V)}(\ ) + x_3 I_{(\mathfrak{e}_7,V)}(\ ) + x_4 I_{(\mathfrak{e}_7,V)}(\ )$
$= (gx_2 + ghx_3 + gsx_4) I_{(\mathfrak{e}_7,V)}(\ ) \Rightarrow kt = x_2 + hx_3 + sx_4$.
This system of equations for t and k can be solved by maple.
The last 3-mesh, , can be resolved by means of a skein relation involving only dashed 3-tangles; one obtains a multiple of .

4-meshes: , , , , , and .
$I_{(\mathfrak{e}_7,V)}(\ ) = x_1 I_{(\mathfrak{e}_7,V)}(\ ) + x_2 I_{(\mathfrak{e}_7,V)}(\ ) + x_3 I_{(\mathfrak{e}_7,V)}(\ ) + x_4 I_{(\mathfrak{e}_7,V)}(\ )$, where the coefficients x_1, x_2, x_3, and x_4 can be determined by applying the skein relations.
The skein relation yields that
$I_{(\mathfrak{e}_7,V)}(\ ) = \frac{1}{\sigma}(I_{(\mathfrak{e}_7,V)}(\ ) - \lambda I_{(\mathfrak{e}_7,V)}(\ ) - \mu s I_{(\mathfrak{e}_7,V)}(\ ) - \rho t I_{(\mathfrak{e}_7,V)}(\ ))$.
We will now derive a planar substitute for . As $V \otimes V$ and $L \otimes L$ have the common direct summands $\mathbf{C}$, L, and U, we can make the following ansatz:
$I_{(\mathfrak{e}_7,V)}(\ ) = X I_{(\mathfrak{e}_7,V)}(\ ) + Y I_{(\mathfrak{e}_7,V)}(\ ) + Z I_{(\mathfrak{e}_7,V)}(\ )$.
First, we determine X and Y:
$fs I_{(\mathfrak{e}_7,V)}(\ ) = f I_{(\mathfrak{e}_7,V)}(\ ) = I_{(\mathfrak{e}_7,V)}(\ ) = X I_{(\mathfrak{e}_7,V)}(\ ) \Rightarrow X = \frac{fs}{c}$
$lt I_{(\mathfrak{e}_7,V)}(\ ) = l I_{(\mathfrak{e}_7,V)}(\ ) = I_{(\mathfrak{e}_7,V)}(\ ) = Y I_{(\mathfrak{e}_7,V)}(\ )$
$= sY I_{(\mathfrak{e}_7,V)}(\ ) \Rightarrow Y = \frac{lt}{s}$.
By means of the following equation, we can substitute :
$I_{(\mathfrak{e}_7,V)}(\ ) = X I_{(\mathfrak{e}_7,V)}(\ ) + Y I_{(\mathfrak{e}_7,V)}(\ ) + Z I_{(\mathfrak{e}_7,V)}(\ )$
$= f^{-1} X I_{(\mathfrak{e}_7,V)}(\ ) + l^{-1} Y I_{(\mathfrak{e}_7,V)}(\ ) + qZ I_{(\mathfrak{e}_7,V)}(\ )$,
and thereby derive the coefficients y_1, y_2, and y_3 in:
$I_{(\mathfrak{e}_7,V)}(\ ) = y_1 I_{(\mathfrak{e}_7,V)}(\ ) + y_2 I_{(\mathfrak{e}_7,V)}(\ ) + y_3 I_{(\mathfrak{e}_7,V)}(\ )$.

For the Lie algebra $\mathfrak{e}_8$ and its adjoint representation L:
Relevant branching types: , , and .
We know how to resolve all 2-meshes and those 3-meshes that do not contain branchings of the form .

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